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On certain canonoid transformations and invariants for the parametric oscillator

Giulio Landolfi and Giulio Soliani

Dipartimento di Fisica dell'Università di Lecce and INFN, Sezione di Lecce, I-73100 Lecce, Italy

E-mail: Giulio.Landolfi@le.infn.it and Giulio.Soliani@le.infn.it

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Abstract
In classical mechanics a transformation in phase space is said to be canonoid if it maps only some Hamiltonian systems into Hamiltonian systems. Once canonoid transformations are considered, these systems can be classically described by means of Lagrangians or Hamiltonians other than the conventional ones. In this context, a basic role is played by the dynamical invariants generated through the Poisson brackets of the new independent variables in phase space. Here we obtain the explicit form of special canonoid transformations of the polynomial type for systems which can be classically described by an equation of the parametric oscillator type and discuss some algebraic properties shown by the associated dynamical invariants.

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1. Introduction

In recent years, interest has been attracted by the theory of dynamical systems admitting alternative formulations (see, e.g., [1–4]). The non-uniqueness of Lagrangians and Hamiltonians associated with a given classical dynamical system may be discussed in connection with the introduction of particular transformations in phase space. As is known, there exist in fact transformations in phase space other than the canonical ones that preserve the Hamiltonian character for the equation as well, but only for a restricted class of Hamiltonians (see, e.g., [5]). They are commonly referred as *canonoid transformations* [5–11][1]. Besides the fact that a relation with the existence of non-Noether symmetries and constants of motion has been pointed out (see, e.g., [3, 11]), a full understanding of the meaning and the properties of canonoid transformations and alternative Lagrangians and Hamiltonians is still missing

[1] In some papers the canonoid transformations have been alternatively called 'canonical transformations' while the mappings preserving the canonical Poisson brackets have been indicated as 'Hamiltonian independent canonical transformations' (see, for instance, [8, 9]).

at the present and efforts have to be done by trying to accumulate significant case studies. The results obtained in [12] concerning polynomial canonoid transformations for generalized time-dependent oscillators provide us a reason of interest which is pertinent to this respect. The aim of this communication is, indeed, to pursue an explicit analysis of dynamical variables for parametric oscillators through canonoid transformations of the type considered in [12]. Precisely, we would like to pay a particular consideration on the algebraic properties, as seen in the original phase space, of constants of motions arising from polynomial canonoid transformations. Our main motivation is that while all the Lagrangians and Hamiltonians which can be obtained generate the same classical equation of motion and give rise to the same basic classical dynamics as the standard parametric oscillator, it is unclear how to exhibit their possible features at the quantum level. On the other hand, it has been discussed in the literature that paying attention to invariants and the underlying *dynamical algebras* may reveal itself as a successful strategy in order to gain a better understanding of properties of dynamical systems and to proceed towards their quantization (see, e.g., [13–17]). After providing explicit formulae for time-dependent coefficients characterizing the canonoid transformations for the parametric oscillator, Poisson brackets among invariants will be scrutinized in the lowest order cases.

The outline of the paper is as follows. In section 2 some basic preliminaries are expounded which regard the concept of canonoid transformation for the parametric oscillator. The formulae for invariants in the cases $n = 2, 3, 4$ (n represents the order defining the polynomial transformations under investigation; see equation (2.9)) are explicitly obtained in section 3. Among the developments inherent to the case $n = 3$, it is worth mentioning a proposition presenting a Lie algebra of the $sl(2, \mathbb{R})$ type whose elements are just the invariants. Section 4 is devoted to a discussion and to concluding comments.

2. Canonoid Hamiltonians for the parametric oscillator and associated invariants

In this section, we shall present polynomial canonoid transformations for parametric oscillators. As we have previously recalled, a transformation in phase space is said canonoid if it maps only some Hamiltonian systems into Hamiltonian ones. In [8] the following theorem relating the possible canonoid transformations of a one-dimensional system has been presented (a generalization to mechanical systems with N degrees of freedom has been discussed in [9]).

Theorem 1. *A mapping $(q, p) \to (Q, P)$ carries a given canonical description of a one-dimensional system into another canonical description if and only if the Poisson bracket $\{Q, P\}_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$ is an invariant (constant of the motion) of the system under consideration: $\frac{\mathrm{d}}{\mathrm{d}t}\{Q, P\}_{q,p} = 0$.*

The problem of obtaining Hamiltonian functions for the parametric oscillator which are linked by a canonoid transformation on phase space can be therefore tackled by resorting to this theorem. As is well known, parametric oscillators are classically described by an equation of motion of the type

$$\ddot{q} + \omega^2(t)q = 0, \tag{2.1}$$

which can be derived from the Lagrangian

$$L_1 = \frac{\dot{q}^2}{2} - \frac{\omega(t)^2 q^2}{2}, \tag{2.2}$$

while the associated Hamiltonian is of the form

$$H_1 = \frac{p^2}{2} + \frac{\omega(t)^2 q^2}{2}, \tag{2.3}$$

where $p = \partial_{\dot q} L_1 = \dot q$ denotes the momentum canonically conjugate to the coordinate q. In [3] it was proven that the Lagrangian function $L_n = L_n(q, \dot q, t)$ expressed by

$$L_n = \sum_{j=0}^{n} \frac{1}{n-j+1} a_j \dot q^{n-j+1} q^j + (\dot a_n - \omega^2 a_{n-1}) \frac{q^{n+1}}{n+1} \tag{2.4}$$

describes the parametric oscillator (2.1) provided that

$$\dot a_0 = -\frac{n-1}{n} a_1, \tag{2.5}$$

$$\dot a_j = (n-j+1)\omega^2 a_{j-1} - (j+1)\frac{n-j-1}{n-j} a_{j+1}, \tag{2.6}$$

with $j = 1, 2, \ldots, n-1$. Indeed, under such hypotheses it turns out that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_n}{\partial \dot q} - \frac{\partial L_n}{\partial q} = I_n(t)[\ddot q + \omega^2 q], \tag{2.7}$$

where $I_n(t)$ is a time-dependent constant of motion (i.e. $\frac{\mathrm{d}I_n}{\mathrm{d}t} = 0$), given by

$$I_n(t) = \frac{\partial^2 L_n}{\partial \dot q^2} = \sum_{j=0}^{n} (n-j) a_j \dot q^{n-j-1} q^j. \tag{2.8}$$

Lagrangians (2.4) basically arise by performing special canonoid transformations on the original parametric oscillator (2.2). They are the transformations $(q, p) \to (Q, P_n)$ on phase space under which the coordinate in configuration space is preserved while $P_n = \frac{\partial L_n}{\partial \dot q}$ is a polynomial of degree n in the variables q and p, i.e.

$$Q = q, \qquad P_n = \sum_{j=0}^{n} a_j p^{n-j} q^j. \tag{2.9}$$

We observe that the relation

$$\{Q, P_n\}_{q,p} = \frac{\partial P_n}{\partial p} = \sum_{j=0}^{n} (n-j) a_j p^{n-j-1} q^j \equiv I_n(t) \tag{2.10}$$

can be deduced (hereinafter, the quantity I_n will be understood as evaluated in the original phase space (q, p), where $\dot q = p$). Hence, the special canonoid transformation where $Q = q$ and P_n defined by (2.9), also named *fouled transformation* in the literature (see references quoted in [6]), satisfies the condition of theorem 1. Before we proceed, we point out that at this stage the functions $a_n(t)$ remain unknown since no functional dependence from the other a_j's ($j = 0, \ldots, n-1$) can be inferred from the compatibility conditions (2.5)–(2.6). This freedom can be exploited to make advantageous choices. (See also the discussion performed in [6].)

To each of the L_n's given by equation (2.4) there corresponds a (canonoid) Hamiltonian $K_n = K_n(Q, P_n)$ which can be derived through the Legendre transformation $K_n = P_n p - L_n$ associated with the mapping $(q, p) \to (Q, P_n)$ of the form (2.9). The dynamical description relying on the adoption of the independent variables Q and P_n should therefore proceed through the analysis of the Hamilton equations

$$\frac{\partial K_n}{\partial P_n} = \dot Q, \qquad \frac{\partial K_n}{\partial Q} = -\dot P_n. \tag{2.11}$$

Here we notice that these canonoid Hamiltonians K_n can be given in the form

$$K_n = \sum_{j=0}^{n} \frac{n-j}{n-j+1} a_j [p(Q, P_n)]^{n-j+1} Q^j - [\dot{a}_n - \omega(t)^2 a_{n-1}] \frac{Q^{n+1}}{n+1} \tag{2.12}$$

where $p(Q, P_n)$ is obtained by inverting the transformation (2.9). Conditions (2.5)–(2.6) ensure indeed that $\frac{\partial K_n}{\partial P_n}$ and $-\frac{\partial K_n}{\partial Q}$ coincide with the time-derivatives of Q and P_n, respectively.

3. Algebraic properties of classical invariants for the canonoid Hamiltonians K_n

In this section, after explicitly deriving the solutions to equations (2.5)–(2.6) defining the canonoid transformations (2.9) for the parametric oscillator, we shall be mainly concerned with algebraic properties of the related invariants. Precisely, we shall discuss in some detail the cases $n = 2, 3, 4$ and we shall focus on the Poisson bracket among distinguished invariants which generally can be cast as (from now on, the Poisson brackets on the phase space (q, p) will be simply denoted as $\{,\}$)

$$\left\{I_n^{(\rho)}, I_n^{(\tau)}\right\} = (n-1) \sum_{k,j=0}^{n} (n-j)(n-k)(j-k) a_j^{(\rho)} a_k^{(\tau)} p^{2n-j-k-3} q^{j+k-1}, \tag{3.1}$$

where $I_n^{(\rho)}$ denotes the invariant constructed in terms of a given set $\left\{a_j^{(\rho)}\right\}$ of independent solutions to equations (2.5)–(2.6). In doing so, we make here the crucial step to recognize that solutions to the system (2.5)–(2.6) are generated by functions a_0 of the type

$$a_0 = \sigma^{n-1} \sum_{j=0}^{\frac{n}{2}-1} \left[\alpha_{n,j+\frac{1}{2}} \cos\theta_{j+\frac{1}{2}}(t) + \beta_{n,j+\frac{1}{2}} \sin\theta_{j+\frac{1}{2}}(t)\right], \tag{3.2}$$

for n even, and

$$a_0 = \sigma^{n-1} \sum_{j=0}^{\frac{(n-1)}{2}} [\alpha_{n,j} \cos\theta_j(t) + \beta_{n,j} \sin\theta_j(t)], \tag{3.3}$$

for n odd, with

$$\theta_\xi(t) = \xi \int^t \frac{\mathrm{d}t'}{\sigma(t')^2}. \tag{3.4}$$

The (generally) time-dependent quantity σ introduced above satisfies the Ermakov equation [18]

$$\ddot{\sigma} + \omega^2 \sigma = \frac{1}{4\sigma^3}. \tag{3.5}$$

Once n is fixed, the general structure (3.2)–(3.3) for a_0 allows us to distinguish n distinct independent solutions $a_0^{(i)}$ each obtained by setting to zero all the $\alpha_{n,j}$, $\beta_{n,j}$ but one (which will be conventionally chosen to be 1). Introduction of each of these $a_0^{(i)}$ into (2.6) and all the other fundamental formulae enables us to obtain n independent invariants, Lagrangians and Hamiltonians. These quantities have some symmetry properties (because of the opposite signs introduced by the derivatives of the sine and the cosine functions). It should be even pointed out that, as long as n increases, higher order derivative terms show up in general formulae for invariants and Hamiltonians which can be handled by properly resorting to the Ermakov equation (3.5) and, eventually, to the Ermakov invariant[2]

$$I_0 = (p\sigma - q\dot{\sigma})^2 + \frac{q^2}{4\sigma^2}. \tag{3.6}$$

Note that when one restricts oneself to the case of a conventional harmonic oscillator, then the Ermakov invariant basically provides the Hamiltonian for the system. Indeed, when $\omega = \lambda =$ const. the solution to equation (3.5) simply reads $\sigma = \frac{1}{\sqrt{2\lambda}}$ so that the quantities $2(\sigma p - \dot{\sigma} q)$ and $\frac{q}{\sigma}$ clearly reduce to $\sqrt{\frac{2}{\lambda}}p$ and $\sqrt{2\lambda}q$, respectively. Hence,

$$H_0 = \tfrac{1}{2}(p^2 + \lambda^2 q^2) = \lambda I_0|_{\omega=\lambda}. \tag{3.7}$$

Before moving to the next section, for the reader's sake we point out that the notation employed hereinafter can be summarized by means of the following two simple prescriptions: (a) objects denoted as $f_{n,\xi}^{(\cos)}$ and $f_{n,\xi}^{(\sin)}$ shall be used to characterize the form assumed by the generic dynamical variable f after it has been evaluated starting from the independent solutions for a_0 of the type $a_{0,n,\xi}^{(\cos)} = \sigma^{n-1}\cos\theta_\xi$ and $a_{0,n,\xi}^{(\sin)} = \sigma^{n-1}\sin\theta_\xi$, respectively; (b) $f_{n,\xi}^{+}$ and $f_{n,\xi}^{-}$ shall be used to denote components of $f_{n,\xi}^{(\cos)}$ along the terms $\cos\theta_\xi$ and $\sin\theta_\xi$, respectively.

3.1. Case $n = 2$

For $n = 2$, equation (2.4) gives

$$L_2 = \frac{1}{3}a_0\dot{q}^3 + \frac{1}{2}a_1 q\,\dot{q}^2 + a_2\dot{q}q^2 + [\dot{a}_2 - \omega^2(t)a_1]\frac{q^3}{3}. \tag{3.8}$$

The classical invariant I_n (see equations (2.8), (2.10)) becomes

$$I_2(t) = 2a_0 p + a_1 q. \tag{3.9}$$

The coefficients a_0 and a_1 are expressed by solutions to equations (2.5) and (2.6), which now read

$$\dot{a}_0 = -\tfrac{1}{2}a_1, \qquad \dot{a}_1 = 2\omega^2(t)a_0. \tag{3.10}$$

So, a_0 obeys the equation for a parametric oscillator with frequency ω, $\ddot{a}_0 + \omega^2 a_0 = 0$. Two independent solutions for a_0 are therefore

$$a_{0,2,\frac{1}{2}}^{(\cos)} = \sigma\cos\theta_{\frac{1}{2}}(t), \qquad a_{0,2,\frac{1}{2}}^{(\sin)} = \sigma\sin\theta_{\frac{1}{2}}(t), \tag{3.11}$$

where σ is defined via (3.5) and $\theta_{\frac{1}{2}}(t) = \frac{1}{2}\int^t \sigma^{-2}\,\mathrm{d}t'$, while the corresponding two independent solutions for a_1 take the form

$$a_{1,2,\frac{1}{2}}^{(\cos)} = \dot{\sigma}\cos\theta_{\frac{1}{2}}(t) - \frac{1}{2\sigma}\sin\theta_{\frac{1}{2}}(t), \qquad a_{1,2,\frac{1}{2}}^{(\sin)} = \dot{\sigma}\sin\theta_{\frac{1}{2}}(t) + \frac{1}{2\sigma}\cos\theta_{\frac{1}{2}}(t). \tag{3.12}$$

It turns out that the invariants in the case $n = 2$ read

$$\begin{pmatrix} I_{2,\frac{1}{2}}^{(\cos)} \\ I_{2,\frac{1}{2}}^{(\sin)} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\frac{1}{2}}(t) & \sin\theta_{\frac{1}{2}}(t) \\ \sin\theta_{\frac{1}{2}}(t) & -\cos\theta_{\frac{1}{2}}(t) \end{pmatrix} \begin{pmatrix} I_{2,\frac{1}{2}}^{+} \\ I_{2,\frac{1}{2}}^{-} \end{pmatrix}, \tag{3.13}$$

[2] The invariant (3.6) for a time-dependent oscillator equation (2.1) has been obtained by Ermakov in [18]. Later, this classical result has been rederived and rediscovered several times by different scientific communities by making use of different methods. For this reason, the quantity (3.6) can be also found referred as the Milne-, Pinney-, or Courant-Snyder invariant (see, e.g., [19, 20] and discussion in [21]).

where

$$I^{+}_{2,\frac{1}{2}} = 2(\sigma p - \dot{\sigma} q), \qquad I^{-}_{2,\frac{1}{2}} = \frac{q}{\sigma}. \tag{3.14}$$

Remark that $\left[I^{(\cos)}_{2,\frac{1}{2}}\right]^2 + \left[I^{(\sin)}_{2,\frac{1}{2}}\right]^2$ basically provides us with the Ermakov invariant:

$$\left[I^{(\cos)}_{2,\frac{1}{2}}\right]^2 + \left[I^{(\sin)}_{2,\frac{1}{2}}\right]^2 = 4I_0. \tag{3.15}$$

If we consider the Poisson brackets among invariants, we straightforwardly recognize that

$$\left\{I^{(\cos)}_{2,\frac{1}{2}}, I^{(\sin)}_{2,\frac{1}{2}}\right\} = 2, \tag{3.16}$$

$$\left\{I^{(\cos)}_{2,\frac{1}{2}}, I_0\right\} = I^{(\sin)}_{2,\frac{1}{2}}, \tag{3.17}$$

$$\left\{I^{(\sin)}_{2,\frac{1}{2}}, I_0\right\} = -I^{(\cos)}_{2,\frac{1}{2}}. \tag{3.18}$$

Hence

$$\left\{I^{(\cos)}_{2,\frac{1}{2}}, I_0\right\}^2 + \left\{I^{(\sin)}_{2,\frac{1}{2}}, I_0\right\}^2 = 4I_0. \tag{3.19}$$

We shall argue later on the generalization of this simple result in cases corresponding to higher values of n.

Let us focus on the case of the standard harmonic oscillator, $\omega(t) = \lambda = \text{const}$. Equation (3.5) now admits the solution $\sigma = \frac{1}{\sqrt{2\lambda}}$ and so one has $\theta_{\frac{1}{2}} = \lambda t$. One can write down the general solution q and the momentum $p = \dot{q}$ of the harmonic oscillator with constant frequency λ in terms of the two invariants following from equation (3.13), i.e.

$$q = \frac{1}{\sqrt{2\lambda}}\left(I^{(\cos)}_{2,\frac{1}{2}} \sin\lambda t - I^{(\sin)}_{2,\frac{1}{2}} \cos\lambda t\right), \qquad p = \sqrt{\frac{\lambda}{2}}\left(I^{(\cos)}_{2,\frac{1}{2}} \cos\lambda t + I^{(\sin)}_{2,\frac{1}{2}} \sin\lambda t\right). \tag{3.20}$$

The conventional Hamiltonian for the time-independent harmonic oscillator can be thus expressed in terms of the invariants according to (see (3.15))

$$H_0 = \frac{1}{2}(p^2 + \lambda^2 q^2) = \frac{\lambda}{4}\left\{\left[I^{(\cos)}_{2,\frac{1}{2}}\right]^2 + \left[I^{(\sin)}_{2,\frac{1}{2}}\right]^2\right\} = \lambda I_0|_{\omega=\lambda}. \tag{3.21}$$

3.2. Case $n = 3$

For $n = 3$, equations (2.4), (2.9) provide us with

$$L_3 = \frac{1}{4}\dot{q}^4 + \frac{1}{3}a_1\dot{q}^3 q + \frac{1}{2}a_2\dot{q}^2 q^2 + a_3\dot{q}q^3 + [\dot{a}_3 - \omega^2(t)a_2]\frac{q^4}{4}. \tag{3.22}$$

From (2.5)–(2.6) we have

$$\dot{a}_0 = -\frac{2}{3}a_1, \qquad \dot{a}_1 = 3\omega^2 a_0 - a_2, \qquad \dot{a}_2 = 2\omega^2 a_1. \tag{3.23}$$

These equations can be manipulated to give $\dddot{a}_0 + 4\omega^2\dot{a}_0 + 4\omega\dot{\omega}a_0 = 0$, whose independent solutions can be written in the form

$$a^{(\cos)}_{0,3,1} = \sigma^2\cos\theta_1(t), \qquad a^{(\sin)}_{0,3,1} = \sigma^2\sin\theta_1(t), \qquad a^{(\cos)}_{0,3,0} = \sigma^2 \tag{3.24}$$

where σ obeys the Ermakov equation (3.5) and $\theta_1 = \int \frac{\mathrm{d}t}{\sigma^2}$. For $n = 3$, equation (2.10) implies $I_3(t) = 3a_0p^2 + 2a_1pq + a_2q^2$. We thus get the following invariants:

$$\begin{pmatrix} I^{(\cos)}_{3,1} \\ I^{(\sin)}_{3,1} \end{pmatrix} = \begin{pmatrix} \cos\theta_1(t) & \sin\theta_1(t) \\ \sin\theta_1(t) & -\cos\theta_1(t) \end{pmatrix}\begin{pmatrix} I^{+}_{3,1} \\ I^{-}_{3,1} \end{pmatrix}, \tag{3.25}$$

with

$$I_{3,1}^{+} = \frac{3}{2\sigma^2}[2\sigma^4(p^2 + q^2\omega^2) - 4pq\dot{\sigma}\,\sigma^3 + q^2(2\ddot{\sigma}\sigma^3 + 2\dot{\sigma}^2\sigma^2 - 1)], \tag{3.26}$$

$$I_{3,1}^{-} = \frac{3q}{\sigma}\,(p\sigma - q\dot{\sigma})\,, \tag{3.27}$$

and

$$I_{3,0}^{(\cos)} = 3[(p^2 + q^2\omega(t)^2)\sigma^2 + q(q\ddot{\sigma} - 2p\dot{\sigma})\sigma + q^2\dot{\sigma}^2] \tag{3.28}$$

(note that $I_{3,0}^{(\cos)} = I_{3,1}^{+} + \frac{3q^2}{2\sigma^2}$). It is worth noting that the Ermakov invariant arises from

$$\left[I_{3,1}^{(\cos)}\right]^2 + \left[I_{3,1}^{(\sin)}\right]^2 = 9\left[(p\sigma - q\dot{\sigma})^2 + \frac{q^2}{4\sigma^2}\right]^2 = 9I_0^2 \tag{3.29}$$

(compare it with equation (3.15)). By focusing on the Poisson brackets

$$\begin{aligned}\left\{I_{3,\xi}^{(i)}, I_{3,\xi'}^{(j)}\right\} &= 18a_{0,3,\xi}^{(i)}\dot{a}_{0,3,\xi'}^{(j)}p^2 + 18\ddot{a}_{0,3,\xi}^{(i)}a_{0,3,\xi'}^{(j)}pq \\ &\quad + 9\dot{a}_{0,3,\xi}^{(i)}\left(\ddot{a}_{0,3,\xi'}^{(j)} + 2\omega^2 a_{0,3,\xi'}^{(j)}\right)q^2 - \begin{pmatrix} i \longleftrightarrow j \\ \xi \longleftrightarrow \xi' \end{pmatrix}\end{aligned} \tag{3.30}$$

($i, j = \sin, \cos$ and $\xi, \xi' = 0, 1$), we are in the position to remark an interesting algebraic property owed by the set of invariants $I_{3,1}^{(\cos)}, I_{3,1}^{(\sin)}, I_{3,0}^{(\cos)}$. Precisely, the following proposition emerges straightforwardly.

Proposition 2. *The invariants* $I_{3,1}^{(\cos)}, I_{3,1}^{(\sin)}, I_{3,0}^{(\cos)}$ *satisfy the* $sl(2, R)$ *algebra*

$$\left\{I_{3,1}^{(\cos)}, I_{3,1}^{(\sin)}\right\} = 6I_{3,0}^{(\cos)}, \qquad \left\{I_{3,1}^{(\sin)}, I_{3,0}^{(\cos)}\right\} = -6I_{3,1}^{(\cos)}, \qquad \left\{I_{3,0}^{(\cos)}, I_{3,1}^{(\cos)}\right\} = -6I_{3,1}^{(\sin)}. \tag{3.31}$$

Before concluding the subsection, note that the Ermakov equation (3.5), as well as the explicit form of the Ermakov invariant I_0, can be exploited to simplify $I_{3,1}^{+}$, and thus $I_{3,0}^{(\cos)}$, according to[3]

$$I_{3,1}^{+} = 3I_0 - \frac{3q^2}{2\sigma^2}, \qquad I_{3,0}^{(\cos)} = 3I_0. \tag{3.32}$$

We point out that such a procedure is crucial, and needs to be applied recursively in fact, in order to have rather compact objects in the higher n cases, where higher order derivatives of σ and higher powers of $\dot{\sigma}$ show up in the formulae.

3.3. Case $n = 4$

In this case we have

$$a_1 = -\tfrac{4}{3}\dot{a}_0, \qquad a_2 = (\ddot{a}_0 + 3\omega^2 a_0), \qquad a_3 = -4\omega\,\dot{\omega}a_0 - \tfrac{14}{3}\omega^2\dot{a}_0 - \tfrac{2}{3}\dddot{a}_0 \tag{3.33}$$

so that

$$I_4 = 4a_0p^3 - 4\dot{a}_0p^2q + 2(\ddot{a}_0 + 3\omega^2 a_0)pq^2 - \left(4\omega\,\dot{\omega}a_0 + \tfrac{14}{3}\omega^2\dot{a}_0 + \tfrac{2}{3}\dddot{a}_0\right)q^3. \tag{3.34}$$

[3] A different way to simplify formulae in concrete analysis may be through the introduction of a function $F(t)$ defined via $F(t) = \dot{\sigma}^2 + \frac{1}{4\sigma^2}$. The function F can be next specified compatibly with equation (3.5) as follows. If $\dot{\sigma} \neq 0$, then we can exploit the relation $\dot{\sigma}^2 + \frac{1}{4\sigma^2} + \omega^2\sigma^2 = F_2(t)$, where $F_2(t) = 2\int \sigma^2\omega\,\dot{\omega}\,\mathrm{d}t$, thus obtaining $F(t) = F_2 - \omega^2\sigma^2$. In the case $\sigma = \text{const.}$ (which is concerned with the case $\omega = \lambda = \text{const}$) the function F should instead be identified as $F = \omega^2\sigma^2 = \lambda/2$.

Insertion of the four independent solutions for a_0, namely

$$a_{0,4,\xi}^{(\cos)} = \sigma^3 \cos\theta_\xi, \qquad a_{0,4,\xi}^{(\sin)} = \sigma^3 \sin\theta_\xi, \tag{3.35}$$

with

$$\theta_\xi(t) = \xi \int^t \frac{\mathrm{d}t'}{\sigma^2}, \qquad \xi = \frac{1}{2}, \frac{3}{2}, \tag{3.36}$$

provides us with four different invariants, $I_{4,\xi}^{(\cos)}$ and $I_{4,\xi}^{(\sin)}$. Precisely, it turns out that

$$\begin{pmatrix} I_{4,\xi}^{(\cos)} \\ I_{4,\xi}^{(\sin)} \end{pmatrix} = \begin{pmatrix} \cos\theta_\xi(t) & \sin\theta_\xi(t) \\ \sin\theta_\xi(t) & -\cos\theta_\xi(t) \end{pmatrix} \begin{pmatrix} I_{4,\xi}^{+} \\ I_{4,\xi}^{-} \end{pmatrix}, \tag{3.37}$$

where

$$\begin{aligned} I_{4,\xi}^{+} &= 4p^3\sigma^3 - 12p^2q\dot\sigma\,\sigma^2 + 6\left[\ddot\sigma + \omega^2\sigma + 2\frac{\dot\sigma^2}{\sigma} - \frac{\xi^2}{3\sigma^3}\right]\sigma^2pq^2 \\ &\quad - 2\left\{\dddot\sigma\sigma^2 + 2\dot\sigma^3 + \left[6\ddot\sigma\sigma + 7\omega^2\sigma^2 - \frac{\xi^2}{\sigma^2}\right]\dot\sigma + 2\omega\,\dot\omega\sigma^3\right\}q^3, \end{aligned} \tag{3.38}$$

$$I_{4,\xi}^{-} = \frac{2q\xi}{3}\left\{6p^2\sigma + \left[7\ddot\sigma + 7\omega^2\sigma + 6\frac{\dot\sigma^2}{\sigma} - \frac{\xi^2}{\sigma^3}\right]q^2 - 12pq\dot\sigma\right\} \tag{3.39}$$

$\left(\xi = \frac{1}{2}, \frac{3}{2}\right)$. Once again, we can take into account the Ermakov equation and the Ermakov invariant thus obtaining (after a cumbersome algebraic manipulation)

$$I_{4,\xi}^{+} = \frac{(p\sigma - q\dot\sigma)}{2}\left[(1-4\xi^2)\frac{q^2}{\sigma^2} + 8I_0\right], \qquad I_{4,\xi}^{-} = \frac{q\xi}{6\sigma}\left[(1-4\xi^2)\frac{q^2}{\sigma^2} + 24I_0\right] \tag{3.40}$$

which easily shows that

$$\left[I_{4,\xi}^{(\cos)}\right]^2 + \left[I_{4,\xi}^{(\sin)}\right]^2 = 16I_0^3. \tag{3.41}$$

From

$$\begin{aligned} \left\{I_{4,\xi}^{(i)}, I_{4,\xi'}^{(j)}\right\} &= 48a_{0,4,\xi}^{(i)}\dot a_{0,4,\xi'}^{(j)}p^4 + 48\ddot a_{0,4,\xi}^{(i)}a_{0,4,\xi'}^{(j)}p^3q \\ &\quad + 24\left[\dot a_{0,4,\xi}^{(i)}\ddot a_{0,4,\xi'}^{(j)} + a_{0,4,\xi}^{(i)}\left(\dddot a_{0,4,\xi'}^{(j)} + 4\omega^2\dot a_{0,4,\xi'}^{(j)}\right)\right]p^2q^2 \\ &\quad + 16\left(\dddot a_{0,4,\xi}^{(i)} + 6\omega\,\dot\omega a_{0,4,\xi}^{(i)}\right)\dot a_{0,4,\xi'}^{(j)}pq^3 \\ &\quad + 4\left[\left(\ddot a_{0,4,\xi}^{(i)} + 3\omega^2\dot a_{0,4,\xi}^{(i)}\right)\left(\dddot a_{0,4,\xi'}^{(j)} + 7\omega^2\dot a_{0,4,\xi'}^{(j)}\right) + 6\omega\,\dot\omega\ddot a_{0,4,\xi}^{(i)}a_{0,4,\xi'}^{(j)}\right]q^4 \\ &\quad - \begin{pmatrix} i \longleftrightarrow j \\ \xi \longleftrightarrow \xi' \end{pmatrix} \end{aligned} \tag{3.42}$$

$(i, j = \sin, \cos$ and $\xi, \xi' = \frac{1}{2}, \frac{3}{2})$, and by exploiting the Ermakov equation as well as the Ermakov invariant, one gets

$$\left\{I_{4,\xi}^{(\cos)}, I_{4,\xi}^{(\sin)}\right\} = 48\xi I_0^2 \tag{3.43}$$

and

$$\begin{pmatrix} \left\{I_{4,\xi}^{(\cos)}, I_0\right\} \\ \left\{I_{4,\xi}^{(\sin)}, I_0\right\} \end{pmatrix} = \begin{pmatrix} \cos\theta_\xi(t) & \sin\theta_\xi(t) \\ \sin\theta_\xi(t) & -\cos\theta_\xi(t) \end{pmatrix} \begin{pmatrix} -\frac{q}{\sigma}\left[(1-4\xi^2)\frac{3q^2}{4\sigma^2} + 8\xi^2I_0\right] \\ \xi(p\sigma - q\dot\sigma)\left[(1-4\xi^2)\frac{q^2}{\sigma^2} + 8I_0\right] \end{pmatrix}. \tag{3.44}$$

Equation (3.44) is more relevant as it appears at a first sight. Let us note, in fact, that by taking the Poisson bracket of equation (3.41) first with $I_{n,\xi}^{(\cos)}$ and next with $I_{n,\xi}^{(\sin)}$, and making use of equation (3.43), we get

$$\left\{I_{4,\xi}^{(\cos)}, I_0\right\} = 2\xi I_{4,\xi}^{(\sin)}, \qquad \left\{I_{4,\xi}^{(\sin)}, I_0\right\} = -2\xi I_{4,\xi}^{(\cos)}, \tag{3.45}$$

respectively. Clearly, the equations (3.44) and (3.45) can be easily shown to be equivalent. Precisely, compatibility between equations (3.44) and (3.45) is achieved provided that

$$(1-4\xi^2)(9-4\xi^2)\frac{q^2}{12\sigma^2}=0.$$

Since the case $n=4$ is concerned with either $\xi=\frac{1}{2}$ or $\xi=\frac{3}{2}$, the above request is therefore satisfied *on shell* (that is, when ξ takes one of the above-mentioned allowed values). We detailed this point because it is instructive about a very typical situation which arises when handling dynamical quantities in the presence of several values allowed for the ξ's.

The algebraic relations among the invariants $I_{4,\xi}^{(\cos)}$, $I_{4,\xi}^{(\sin)}$ and I_0 can be therefore expressed by equations (3.43) and (3.45). Remark that the latter implies

$$\left\{I_{4,\xi}^{(\cos)},I_0\right\}^2+\left\{I_{4,\xi}^{(\sin)},I_0\right\}^2=64\xi^2I_0^3. \tag{3.46}$$

4. Discussion

In this paper, fouled polynomial canonoid transformations (2.9) for the parametric oscillators have been investigated. After analysing the basic structure of solutions for the differential system for the time-dependent coefficients $a_j(t)$, some invariants have been obtained. Their explicit knowledge enabled us to depict features which could not have been noticed otherwise. Basically, we point out how the coefficients a_j are related to the dynamics of the 'original' parametric oscillator, and thus to the associated Ermakov equation. The presence of a link is somehow expected by construction, in principle. Interestingly, we have seen that, as long as the order n of the polynomial canonoid transformation (2.9) increases, the net consequence is that the time-dependent amplitude of each independent solution for a_0 changes according to powers of order $n-1$ of the time-dependent amplitude of the original parametric oscillator while the corresponding time-dependent phases differ by a proper proportionality factor (equation (3.2)–(3.3)). For $n=2$, the differential equation for a_0 is just that of a parametric oscillator with the same time-dependent frequency as the original one (see equation (2.1) and (3.10)). A consequence of the fact that the general structure for a_0 (and, hence, for the a_j with $j=1,\ldots,n-1$) is a linear combination of time-dependent sine and cosine terms is that, once the dynamical invariants (2.10) (which depend linearly from these time-dependent functions) are considered, one sees, in a sense, time-dependent rotation matrices acting on some basic objects. In the cases of the Lagrangians a term linear (depending on $p,q,\sigma,\dot\sigma$) with respect to $\dot a_n$ also enters in the matter. When a_n is chosen to be either constant or $a_n=a_n(a_0,\ldots,a_{n-1})$ (and hence $a_n=a_n(a_0)$), clearly even the alternative Lagrangians L_n can be seen in terms of a particular time-dependent rotation. The explicit form of Lagrangians and, especially, invariants have been given when $n=2,3,4$. We have also investigated the Poisson brackets among invariants attempting to express formulae in a meaningful form by letting the role of the basic Ermakov invariant I_0 be as more explicit as possible. Kind of clarification about the meaning of the rotation matrix arises in terms of objects defined by suitable powers of the basic Ermakov invariant I_0. Equations (3.15)–(3.19), (3.29), (3.31), (3.32), (3.41)–(3.46) show indeed that, when $n=2,3,4$, one has

$$\left[I_{n,\xi}^{(\cos)}\right]^2+\left[I_{n,\xi}^{(\sin)}\right]^2=n^2I_0^{n-1},\qquad \left\{I_{n,\xi}^{(\cos)},I_{n,\xi}^{(\sin)}\right\}=(n-1)n^2\xi I_0^{n-2}, \tag{4.1}$$

$$\left\{I_{n,\xi}^{(\cos)},I_0\right\}=2\xi I_{n,\xi}^{(\sin)},\qquad \left\{I_{n,\xi}^{(\sin)},I_0\right\}=-2\xi I_{n,\xi}^{(\cos)}, \tag{4.2}$$

and therefore,

$$\left\{I_{n,\xi}^{(\cos)},I_0\right\}^2+\left\{I_{n,\xi}^{(\sin)},I_0\right\}^2=4n^2\xi^2I_0^{n-1}. \tag{4.3}$$

It would be rather tempting to conjecture that such relations hold in wider generality. We checked that they hold even in the case $n = 5$. In such a case, a direct computation also shows that

$$I^{+}_{5,\xi} = 5I_0^2 - 5\xi^2 \frac{q^2}{2\sigma^2}\left[(1-\xi^2)\frac{q^2}{12\sigma^2} + I_0\right], \tag{4.4}$$

$$I^{-}_{5,\xi} = 5\xi \frac{q}{\sigma}\left[(1-\xi^2)\frac{q^2}{6\sigma^2} + I_0\right]. \tag{4.5}$$

Hence, the invariant associated to the case $\xi = 0$ is $I_{5,0} = I^{(\cos)}_{5,0} = I^{+}_{5,0} = 5I_0^2$. We thus wonder if a relation of the type

$$I_{n,0} \doteqdot I^{(\cos)}_{n,0} = nI_0^{(n-1)/2} \tag{4.6}$$

does generally hold when dealing with n odds and, in case, if it could have been deduced alternatively.

The relevance of the proposition 2 (see equation (3.31)) relies on the fact that it characterizes a dynamical algebra underlying the case $n = 3$. The result should be of interest when moving towards the direction of quantization of the fouled canonoid Hamiltonians for the parametric oscillator. Once alternative Hamiltonians for the parametric oscillators are explicitly constructed thanks to (3.2)–(3.3), the forthcoming step in order would consist in the study of their quantum mechanical aspects. Nevertheless, the possibility of handling dynamical systems through alternative classical Lagrangians and, correspondingly, through alternative classical Hamiltonians may open new scenarios and pose some questions. Clearly everything is straight at the classical level, since alternative Lagrangians and Hamiltonians give rise to the same equation of motion obtained 'conventionally'. But at the quantum level one may even wonder whether alternative Hamiltonians actually mimic dynamical systems with some characteristics different from those endowed with standard ones. Shedding light on cases regarding quantum oscillators represents indeed a step both due, because of their relevance in the realm of physics, and useful, because of the feasibility of concrete experimental tests. Nevertheless, this issue goes beyond our present scopes and will be considered elsewhere.

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